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On the Statistical Mechanics Approach in the Random Matrix Theory: Integrated Density of States

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We consider the ensemble of random symmetric $n \times n$ matrices specified by an orthogonal invariant probability distribution. We treat this distribution as a Gibbs measure of a mean-field-type model. This allows us to show that the normalized eigenvalue counting function of this ensemble converges in probability to a nonrandom limit as $n \to \infty$ and that this limiting distribution is the solution of a certain self-consistent equation.

KEY WORDS: Random matrix; integrating density of states; statistical mechanics; mean field-theory.

1. INTRODUCTION, MAIN RESULTS, AND EXAMPLES

Consider the ensemble of random real and symmetric $n \times n$ matrices M defined by the probability distribution

$$p_n(M)\, dM = (Z_n^{(0)})^{-1} \exp[-n \operatorname{Tr} V(M)] \prod_{i \leqslant j} dM_{ij} \tag{1.1}$$

where $Z_n^{(0)}$ is the normalization factor and $V(\lambda)$, $\lambda \in \mathbf{R}$, is a real-valued function satisfying the following conditions:

(a) $V(\lambda)$ is bounded below.

(b) $V(\lambda) \geqslant (2+\varepsilon) \ln |\lambda|$ for some $\varepsilon > 0$, if $|\lambda|$ is large enough.

(c) There exists $\gamma > 0$ such that for any $L > 0$

$$|V(\lambda_1) - V(\lambda_2)| \leqslant C(L)\, |\lambda_1 - \lambda_2|^\gamma \qquad \text{if} \quad |\lambda_1|, |\lambda_2| \leqslant L \tag{1.2}$$

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We will call ensemble (1.1) the generalized orthogonal ensemble because its density with respect to the "Lebesgue" measure dM is obviously orthogonal invariant. The simplest example of this ensemble is the Gaussian orthogonal ensemble (GOE) corresponding to $V(\lambda) = \lambda^2/4a^2$. In this case the entries M_{ij}, $i \leqslant j$, are independent Gaussian random variables, which allow explicit calculation of many important spectral characteristics of the GOE (see, e.g., ref. 2 and references therein). One of the simplest characteristics that is rather interesting in many respects is the normalized eigenvalue counting function

$$N_n(\lambda) = n^{-1} \sum_{\lambda_i < \lambda} 1 \tag{1.3}$$

The limit of the expectation $\mathbf{E}\{N_n(\lambda)\}$ of (1.3) with respect to the distribution (1.1) for $n \to \infty$ is called the integrated density of states (IDS) and is denoted by $N(\lambda)$:

$$\lim_{n \to \infty} \mathbf{E}\{N_n(\lambda)\} = N(\lambda)$$

According to Wigner,[1] for the GOE [i.e., for the (1.1) with $V(\lambda) = \lambda^2/4a^2$] $N(\lambda)$ exists and its derivative $\rho(\lambda)$ is

$$N'(\lambda) = \rho(\lambda) = \begin{cases} (2\pi a^2)^{-1} (4a^2 - \lambda^2)^{1/2} & \text{if } |\lambda| \leqslant 2a \\ 0 & \text{otherwise} \end{cases} \tag{1.4}$$

This limiting eigenvalue distribution is known as the semicircle (Wigner) law. Since Wigner's paper[1] appeared, numerous aspects of the random matrix theory have been extensively developed and used in probability theory, nuclear physics, quantum chaology, quantum field theory, and statistical mechanics (see refs. 2–5 and references therein). However, most of the rigorous results in this field concern the GOE or other ensembles with statistically independent although not necessarily Gaussian entries for $i \leqslant j$. In the latter case the joint probability distribution of an ensemble is not invariant with respect to the transformations $M \to OMO^*$, $O \subset O(n)$. On the other hand, the distribution (1.1) is orthogonal invariant for any $V(\lambda)$, but if $V(\lambda) \neq \lambda^2/4a^2$ for some $a > 0$, then (1.1) implies rather strong statistical dependence between the entries M_{ij}, $i \leqslant j$. The study of a unitary invariant analog of (1.1) [i.e., the case when M's are Hermitian matrices and $V(\lambda)$ is an even polynomial with nonnegative coefficients] was started in the important physical papers[6, 7] motivated by quantum chromodynamics. In recent years similar ensembles with polynomial $V(\lambda)$ have been actively studied in connection with a certain nonperturbative approach in bosonic string theory and two-dimensional gravity (see refs. 5 and 15 and references therein).

In this paper we show rigorously that under conditions (1.2) the random eigenvalue distribution (1.3) converges in probability to the non-random limiting IDS as $n \to \infty$. We also give a method of calculation of the IDS and use this method to consider some examples of the IDS, including the explicit form of the IDS for some polynomial V's (these forms were found by different methods in refs. 7–9 and 17). More detailed study of the properties of the IDS of ensemble (1.1) as well as analogous results for some other ensembles will be published elsewhere.

Our starting point is the joint probability distribution of the eigenvalues corresponding to (1.1),

$$p_n(\lambda_1,..., \lambda_n) = Q_n^{-1} \exp\left\{-n \sum_{i=1}^{n} V(\lambda_i)\right\} \prod_{i<j} |\lambda_i - \lambda_j| \tag{1.5}$$

(see, e.g., ref. 2). According to Wigner[1] (see also refs. 6 and 10), (1.5) can be rewritten as the canonical Gibbs distribution

$$p_n(\lambda_1,..., \lambda_n) = Q_n^{-1} \exp\{-nH_n(\lambda_1,..., \lambda_n)\} \tag{1.6}$$

corresponding to a one-dimensional system of n particles with the Hamiltonian

$$H_n(\lambda_1,..., \lambda_n) = \sum_{i=1}^{n} V(\lambda_i) - \frac{1}{n} \sum_{i<j} \ln |\lambda_i - \lambda_j| \tag{1.7}$$

at the temperature n^{-1}. The first term of the r.h.s. is analogous to the energy of particles due to the external field $V(\lambda)$ and the second term is analogous to the interaction (Coulomb) energy.

This statistical mechanical interpretation of the density (1.5) was considerably developed and efficiently used by Dyson.[10]

It is important that the density (1.6) and the Hamiltonian (1.7) contain explicitly the "number of particles" n. This allows us to regard (1.6) and (1.7) as analogs of molecular field models of statistical mechanics. This analogy was used in physical papers.[1,6] The rigorous treatment of the molecular field models of a rather general form was given by several authors (see, e.g., refs. 10–13). In particular, refs. 13 and 14 contain an approach whose extension allows us to carry out a rigorous analysis of the Hamiltonian (1.6). The result of this analysis is as follows:

Theorem 1. Let the ensemble of random matrices be specified by (1.1) in which a real-valued function $V(\lambda)$ satisfies condition (1.2). Then the normalized eigenvalue counting function (1.3) corresponding to this ensemble converges in probability to the nonrandom absolutely continuous IDS $N(\lambda)$ whose density is uniquely determined by the conditions

$$\rho(\lambda) \geqslant 0 \tag{1.8}$$

$$\int \rho(\lambda)\, d\lambda = 1 \tag{1.9}$$

$$-\int \ln |\lambda_1 - \lambda_2|\, \rho(\lambda_1)\, \rho(\lambda_2)\, d\lambda_1\, d\lambda_2 < \infty \tag{1.10}$$

the function

$$u(\lambda) = \int \ln |\lambda - \lambda'|\, \rho(\lambda')\, d\lambda' - V(\lambda) \tag{1.11}$$

is bounded from above, and

$$\operatorname{supp} \rho(\lambda) \subset \{\lambda \colon u(\lambda) = \max_{\lambda'} u(\lambda')\} \tag{1.12}$$

Theorem 1 will be proved in the next two sections. Here we give several remarks and examples.

Remarks. (i) The random matrix theory considers also the ensembles of Hermitian and so-called quaternion real random matrices whose probability densities are invariant under unitary and symplectic transformations, respectively (the Gaussian cases of these ensembles are known as GUE and GSE). The eigenvalue probability densities of these ensembles differ from (1.1) in the powers of the factor $|\Delta(\lambda_1,..., \lambda_n)|$, where

$$\Delta(\lambda_1,..., \lambda_n) = \prod_{i<j} (\lambda_i - \lambda_j) \tag{1.13}$$

This power is 2 for the unitary ensemble and 4 for the simplectic ensemble. Therefore the general form of the joint eigenvalue probability distribution for all three ensembles is

$$p_{n\beta}(\lambda_1,..., \lambda_n) = Q_{n\beta}^{-1} \exp\left\{-n \sum_{i=1}^{n} V(\lambda_i)\right\} \prod_{i<j} |\lambda_i - \lambda_j|^{\beta} \tag{1.14}$$

where $\beta = 1$, 2, 4 for the orthogonal, unitary, and symplectic ensembles, respectively.

Our main result given by Theorem 1 is valid for $\beta = 2$, 4 as well if we introduce the factor β in front of the integral of the r.h.s. of (1.11). The analogs of formula (1.7) in these cases have the factor β in front of the double sums, i.e., β plays the role of a coupling constant of the respective n-particle system.

(ii) Relation (1.12) is just the zero-temperature case of the self-consistent equation for the particle density and is well known in molecular field theory. Indeed, we have mentioned before that the large parameter n plays different roles in formulas (1.6) and (1.7). In the former, n plays the role of the inverse temperature, while in the latter, the factor n^{-1} allows us to treat (1.7) as a molecular field-type Hamiltonian. Thus, if the factor n in (1.6) were replaced by the inverse temperature $(kT)^{-1}$, then the arguments which we used to prove Theorem 1 would lead to the standard molecular field equation for the particle density

$$\rho(\lambda) = \frac{\exp\{-(kT)^{-1} u(\lambda)\}}{\int \exp\{-(kT)^{-1} u(\mu)\} d\mu}$$

[in fact a similar equation will appear below; see (2.11)]. Now, if in this equation we perform the limiting transition $T \to 0$, we obtain (1.12).

Thus, from the statistical mechanical point of view, Theorem 1 asserts that the zero-temperature case of the molecular field equation for the model (1.7) can be obtained not only after subsequent limiting transitions $n \to \infty$ and then $T \to 0$, but also as a result of simultaneous limiting transitions $n \to \infty$, $T \to 0$, provided that the product nT is fixed.

(iii) By the method of Theorem 1 (see Proposition in the next section) one can also show that the expression for the ground-state energy of the statistical mechanical model (1.6), i.e., $E = -\lim_{n \to \infty} n^{-2} \ln Q_n$, has the following form:

$$E = -\frac{\beta}{2} \int \ln |\lambda - \lambda'| \, \rho(\lambda) \, \rho(\lambda') \, d\lambda \, d\lambda' + \int \rho(\lambda) \, V(\lambda) \, d\lambda \qquad (1.15)$$

where $\rho(\lambda)$ is given by Theorem 1. Moreover, in total agreement with statistical mechanics, E can be obtained as the minimum value of the "electrostatic" energy (cf. results of ref. 6)

$$E = \min_{\nu} \left\{ -\frac{\beta}{2} \int \ln |\lambda - \lambda'| \, \nu(d\lambda) \, \nu(d\lambda') + \int V(\lambda) \, \nu(d\lambda) \right\}$$

of two-dimensional (line) charges whose distribution on the real line is described by the measure $\nu(\cdot)$, $\nu(R) = 1$. Then Theorem 1 implies that under its conditions a minimizing measure has the density $\rho(\lambda)$ satisfying

(1.8)–(1.11). This density is the unique solution of the extremum equation of the variational problem (1.12):

$$\beta \int_{\operatorname{supp} \rho} \ln |\lambda - \lambda'| \, \rho(\lambda') \, d\lambda' = V(\lambda) + \text{const}, \qquad \lambda \in \operatorname{supp} \rho \tag{1.16}$$

If we differentiate (1.16) with respect to λ, we obtain the singular integral equation for $\rho(\lambda)$:

$$\beta \int_{\operatorname{supp} \rho} \frac{\rho(\lambda') \, d\lambda'}{\lambda - \lambda'} = V'(\lambda), \qquad \lambda \in \operatorname{supp} \rho \tag{1.17}$$

This equation has a simple electrostatic interpretation: it is just the equilibrium condition for the continuously distributed charges of strength $\beta^{1/2}$ subjected to the external electrostatic potential $V(\lambda)$ [electric force $-V'(\lambda)$]. This equation appeared for the first time in Wigner's work[1] and afterward was used in numerous works in this field (see, e.g., 6–8, 10, 15, and 17). This equation allows us to find $\rho(\lambda)$ and its support in a number of interesting cases (see examples below and also in refs. 6–8, 15, and 18).

(iv) Repeating almost literally the arguments which we used to prove Theorem 1, we can also prove an analogous result for a more general ensemble of random matrices with an orthogonal invariant density:

$$p_n(M) = (Z_n^{(0)})^{-1} \exp[-n V_n(\lambda_1, \ldots, \lambda_n)] \tag{1.18}$$

where the function V_n is

$$V_n(\lambda_1, \ldots, \lambda_n) = \sum_{i=1}^{n} V(\lambda_i) + \sum_{k=2}^{n} \frac{1}{k! \, n^{(k-2)}} \sum_{i_1 \neq i_2 \neq \cdots \neq i_k} V^{(k)}(\lambda_{i_1}, \ldots, \lambda_{i_k})$$

with bounded, symmetric, and Hölder continuous functions $V^{(k)}$, $k = 2, 3, \ldots$, satisfying the following condition. The functional

$$U(c) = \sum_{k=2}^{n} \frac{1}{k!} \int V^{(k)}(\lambda_1, \ldots, \lambda_{2k}) \prod_{i=1}^{k} c(\lambda_i) \, d\lambda_i \tag{1.19}$$

is convex in the space of smooth functions with compact supports.

Theorem 2. Let the ensemble of random matrices be specified by (1.18) in which a real-valued function $V(\lambda)$ satisfies condition (1.2). Then the normalized eigenvalue counting function (1.3) corresponding to this ensemble converges in probability to the nonrandom absolutely continuous

IDS $N(\lambda)$ whose density is uniquely determined by conditions (1.8)–(1.10) and (1.12), where now $u(\lambda)$ is [cf. (1.11)]

$$u(\lambda) = \int \ln |\lambda - \lambda'| \, \rho(\lambda') \, d\lambda' - V(\lambda) - \sum_{k=2}^{\infty} \frac{1}{k!} \int V_k(\lambda, \lambda_1, \ldots, \lambda_k) \prod_{i=1}^{k} \rho(\lambda_i) \, d\lambda_i \tag{1.20}$$

and as before has to be bounded from above.

We mention here two examples where condition (1.19) is satisfied. The first one corresponds to $V^{(k)} = 0$, $k \geqslant 3$, and $V^{(2)}(\lambda_1, \lambda_2)$ defining a positive operator in the space $L_2(-l, l)$, where l is large enough. In particular, if $F(\lambda) \in L_1(R)$ has a nonnegative Fourier transform, then we take $V^{(2)}(\lambda_1, \lambda_2) = F(\lambda_1 - \lambda_2)$. In the second example we take the sequence $\{V^{(k)}\}_{k=2}^{\infty}$ to be a sequence of moments of some random process $\xi(\lambda)$, $\lambda \in R$: $V^{(k)}(\lambda_1, \ldots, \lambda_k) = \mathbf{M}\{\xi(\lambda_1), \ldots, \xi(\lambda_k)\}$, $k = 2, 3, \ldots$, where the symbol $\mathbf{M}\{\cdots\}$ denotes the mathematical expectation with respect to this random process. We assume that the generating functional

$$\mathbf{M}\left\{\exp\left\{\int \xi(\lambda) \, c(\lambda) \, d\lambda\right\}\right\}$$

exists for any smooth function $c(\lambda)$ with a compact support. It is easy to see that both examples satisfy condition (1.19).

(v) Our results imply that the first correlation function of the statistical mechanical system with Hamiltonian (1.7) satisfies the self-consistent (molecular field) equation (1.12) in the limit $n = \infty$ and the second (and every higher) correlation function is a product of the first correlation function in this limit. These facts are also in full agreement with statistical mechanics, according to which in a molecular field model there is no correlation between particles at distinct points (see, e.g., ref. 13). However, the question of primary interest in the random matrix theory is the behavior of correlation functions for interparticle distances of the order $1/n$. In particular, the limit of the second irredubible correlation function $\rho_n(\lambda_1, \lambda_2) - \rho(\lambda_1)\, \rho(\lambda_2)$ for $\lambda_{1,2} = \lambda_0 + \xi_{1,2}/N\rho_n(\lambda)$ determines the probability distribution of nearest-neighbor eigenvalue spacings, which is of considerable interest in statistical nuclear physics[2] and quantum chaology.[4] According to the universality conjecture,[2] the form of this limit does not depend on a concrete ensemble [i.e., on $V(\lambda)$ in (1.1)] and depends only on β in (1.14). The proofs of this hypothesis for a variety of concrete ensembles are given in refs. 2, 9, and 10.

Examples. (i) $V(\lambda)$ is a twice differentiable convex function, i.e., $V''(\lambda) \geqslant 0$. In this case it follows from (1.11) that $u(\lambda) + V(\lambda)$ is concave for all $\lambda, \lambda \notin \operatorname{supp} \rho$. Thus, according to (1.12), the support of $\rho(\lambda)$ is an interval (a, b). Besides it can be shown that, under our conditions on $V(\lambda)$, $\rho(\lambda)$ is a bounded function for $a \leqslant \lambda \leqslant b$. Then, according to ref. 18, the singular integral equation (1.17) has a unique bounded solution:

$$\rho(\lambda) = \frac{1}{\beta\pi^2} [(b-\lambda)(\lambda-a)]^{1/2} \int_a^b \frac{V'(\mu)\, d\mu}{(\lambda-\mu)[(b-\mu)(\mu-a)]^{1/2}} \qquad \lambda \in (a, b) \tag{1.21}$$

provided that

$$\int_a^b \frac{V'(\mu)\, d\mu}{[(b-\mu)(\mu-a)]^{1/2}} = 0 \tag{1.22}$$

This condition and the normalization condition (1.9) determine both endpoints a and b. If $V(\lambda)$ is an even function, then $a = -b$, (1.22) is trivial, and b is determined by (1.9).

Using the identity

$$\int_a^b \frac{d\mu}{(\lambda-\mu)[(b-\mu)(\mu-a)]^{1/2}} = 0, \qquad \lambda \in (a, b) \tag{1.23}$$

we can rewrite (1.21) in the manifestly positive form

$$\rho(\lambda) = \int_a^b \frac{(V'(\mu) - V'(\lambda))\, d\mu}{(\lambda-\mu)[(b-\mu)(\mu-a)]^{1/2}} \tag{1.24}$$

(ii) $V(\lambda) = |\lambda|^\alpha$. For $\alpha > 1$, $\beta = 2$ this ensemble was studied in ref. 9 by the orthogonal polynomial technique.[2] It was proved that $\rho(\lambda)$ has the form

$$\rho(\lambda) = B_\alpha^{-1} v_\alpha(B_\alpha^{-1} \lambda) \tag{1.25}$$

where

$$B_\alpha^\alpha = 2\alpha\pi^{-1} \int_0^1 t^\alpha (1-t^2)^{-1/2}\, dt$$

$$v_\alpha(t) = \begin{cases} \alpha\pi^{-1} \int_{|t|}^1 \tau^{\alpha-1}(\tau^2 - t^2)^{-1/2}\, d\tau & \text{if } |t| \leqslant 1 \\ 0 & \text{otherwise} \end{cases}$$

It is easy to check that expression (1.25) satisfies conditions (1.8)–(1.12) for all $\alpha > 0$. Therefore on the basis of Theorem 1 we conclude that $\rho(\lambda)$ has the form (1.15) for all positive α. Notice that for $0<\alpha<1$ this density is unbounded at zero. For $\alpha>1$, (1.25) can also be obtained by solving Eq. (1.17).

(iii) $V(\lambda)=(\lambda^4/4)+m^2(\lambda^2/2)$. This ensemble was studied in refs. 6 and 7, motivated by quantum field theory (see also refs. 15 and 17). If $m^2\geqslant 0$, then $V''(\lambda)\geqslant 0$, and we can use (1.24) with $a=-b$ to find ρ explicitly:

$$\rho(\lambda)=\frac{1}{\pi}(a^2-\lambda^2)^{1/2}\left(\lambda^2+m^2+\frac{a^2}{2}\right) \tag{1.26}$$

It is easy to check that this formula satisfies (1.8)–(1.12) if $m^2\geqslant -2^{-3/2}$. But for $m^2<-2^{-3/2}$ the r.h.s. of (1.26) is negative in a certain neighborhood of the origin. Therefore it is natural to look for the solution of (1.17), whose support consists of two symmetric intervals, $(-b, -a)$ and (a, b). By using the respective formulas from ref. 18, it is easy to find that for $m^2<-2^{-3/2}$

$$\rho(\lambda)=\pi^{-1}\,|\lambda|\,[(b^2-\lambda^2)(\lambda^2-a^2)]^{1/2}, \qquad a^2\leqslant\lambda^2\leqslant b^2$$

provided that

$$\int_a^b \frac{V'(\mu)\,\mu\,d\mu}{[(b^2-\mu^2)(\mu^2-a^2)]^{1/2}}=0 \tag{1.27}$$

This condition and normalization condition (1.9) determine the endpoints a and b: $a^2=-\sqrt{2}-m^2$, $b^2=\sqrt{2}-m^2$.

(iv) $V(\lambda)=(\lambda^6/6)+g(\lambda^4/4)+m^2(\lambda^2/2)$. In this case support $\rho(\lambda)$ consists of one, two, or three intervals. The former two cases can be studied by a method analogous to that used in the previous example. The latter (three-intervals) case corresponds to $V(\lambda)$ having three well-pronounced wells ($g\ll -1$, $|m^2|\gg 1$) and requires additional arguments. Indeed, for an even r.h.s. in (1.17) the condition of unique solubility of this equation in the class of bounded nonnegative functions supported on three intervals $S=\{\lambda\colon \lambda^2\leqslant a^2,\ b^2\leqslant\lambda^2\leqslant c^2,\ 0<a<b<c<\infty\}$ is[19]

$$\int_S \frac{\lambda V'(\lambda)\,d\lambda}{[X(\lambda)]^{1/2}}=0 \tag{1.28}$$

where $X(\lambda)=(a^2-\lambda^2)(\lambda^2-b^2)(\lambda^2-c^2)$ [cf. (1.27)]. Thus, we have two equations [(1.28) and (1.9)] to find three endpoints. The statistical mechanical (or electrostatic) meaning of this fact is simple: we can fix arbitrarily charges ρ_1 and ρ_2 of central and both additional wells provided that $\rho_1 + 2\rho_2 = 1$. This yields a one-parameter family of charge distributions. We obtain this third condition allowing us to determine uniquely all endpoints if we use Eq. (1.12), which implies $u(a) = u(b)$, or if we minimize the energy (1.15).

2. PROOF OF THEOREM 1

To study $N_n(\lambda)$, we will introduce its Stieltjes transform

$$g_n(z) = \int \frac{dN_n(\lambda)}{z - \lambda}, \qquad \Im z \neq 0 \tag{2.1}$$

and prove that $g_n(z)$ has a nonrandom limit. To this end it is sufficient to prove the two following statements. The first one is the self-averaging property of $g_n(z)$, i.e.,

$$d_n(z) = \mathbf{E}\{|g_n(z) - \mathbf{E}\{g_n(z)\}|^2\} \to 0, \qquad n \to \infty$$

and the second is the existence of the limit $\mathbf{E}\{g_n(z)\}$ for $n \to \infty$. In fact we prove that

$$d_n(z) \leqslant \text{const} \cdot \frac{\ln n}{n} \tag{2.2}$$

It is easy to see that $d_n(z)$ is

$$\begin{aligned} d_n(z) = {} & \frac{2}{n^2} \sum_{i<j} \int (\rho_n(\lambda_1, \lambda_2) - \rho_n(\lambda_1)\, \rho_n(\lambda_2)) \frac{d\lambda_i}{z - \lambda_i} \cdot \frac{d\lambda_j}{\bar{z} - \lambda_j} \\ & + \frac{1}{n^2} \sum_{i=1}^{n} \int \rho_n(\lambda_i) \left| \frac{1}{z - \lambda_i} - \mathbf{E}\{g_n(z)\} \right|^2 d\lambda_i \end{aligned} \tag{2.3}$$

where

$$\mathbf{E}\{g_n(z)\} = \int \rho_n(\lambda) \frac{d\lambda}{z - \lambda} \tag{2.4}$$

and $\rho_n(\lambda_1),..., \rho_n(\lambda_1,..., \lambda_k)$ are the correlation functions of the Hamiltonian (1.7), i.e.,

$$\rho_n(\lambda_1,..., \lambda_k) = Q_n^{-1} \int \exp\{-nH(\lambda_1,..., \lambda_n)\}\, d\lambda_{k+1} \cdots d\lambda_n, \quad k = 1, 2,... \tag{2.5}$$

where Q_n is the same as in (1.5). By virtue of Lemma 1, proved in Section 3, to find the limits of $\rho_n(\lambda_1,..., \lambda_k)$ for $n \to \infty$ it suffices to restrict our considerations to a certain finite interval $(-L, L)$. To simplify the notations, let us change the variables in formulas (2.3)–(2.5),

$$x_i = \lambda_i/2L; \qquad v(x) = V(2xL)$$

We obtain the Hamiltonian

$$H(x_1,..., x_n) = \sum_{i=1}^{n} v(x_i) - \frac{1}{n} \sum_{i<j} \ln |x_i - x_j|$$

in which $|x_i| \leqslant 1/2$, $i = 1,..., n$. We are interested in the large-n behavior of [cf. (2.5)]

$$\rho_n(x_1,..., x_k) = Z_n^{-1} \int_{-1/2}^{1/2} \exp\{-nH(x_1,..., x_n)\}\, dx_{k+1} \cdots dx_n, \qquad k = 1, 2,... \tag{2.6}$$

with

$$Z_n = \int_{-1/2}^{1/2} \exp\{-nH(x_1, x_2,..., x_n)\}\, dx_1 \cdots dx_n$$

We will analyze this behavior by using a certain modification of the method of study of mean-field models proposed in ref. 13. For any function $c(x)$ satisfying the inequality

$$-\int_{-1/2}^{1/2} \ln |x-y|\, c(x)\, c(y)\, dx\, dy < \infty \tag{2.7}$$

introduce the "approximating" Hamiltonian

$$\begin{aligned} H_a(x_1,..., x_n; c) = \sum_{i=1}^{n} v(x_i) - \sum_{i=1}^{n} \int_{-1/2}^{1/2} \ln |x_i - y|\, c(y)\, dy \\ + \tfrac{1}{2}(n+1) \int_{-1/2}^{1/2} \ln |x-y|\, c(x)\, c(y)\, dx\, dy \end{aligned} \tag{2.8}$$

Let

$$\begin{aligned} \Psi_n(c) &= \frac{1}{n^2} \ln \int_{-1/2}^{1/2} \exp\{-nH_a(x_1,..., x_n; c)\}\, dx_1 \cdots dx_n \\ &= -\frac{n+1}{2n} \int_{-1/2}^{1/2} \ln |x-y|\, c(x)\, c(y)\, dx\, dy \\ &\quad + \frac{1}{n} \ln \int_{-1/2}^{1/2} dx \exp\left\{-nv(x) + n \int_{-1/2}^{1/2} \ln |x-y|\, c(y)\, dy\right\} \end{aligned} \tag{2.9}$$

Then by the Bogoliubov inequality

$$R \leqslant \Psi_n(c) - \frac{1}{n^2} \ln Z_n \leqslant R_a \tag{2.10}$$

where

$$R(c) = (nZ_n)^{-1} \int_{-1/2}^{1/2} (H - H_a) \exp\{-nH\} \, dx_1 \cdots dx_n$$

and

$$R_a(c) = (n)^{-1} \exp\{-n^2 \Psi_n(c)\} \int_{-1/2}^{1/2} (H - H_a)$$

$$\times \exp\{-nH_a(x_1,..., x_n; c)\} \, dx_1 \cdots dx_n$$

Using the fact that H and H_a are symmetric with respect to the x_i, one can rewrite R as follows:

$$R(c) = -\frac{n-1}{2n} \int_{-1/2}^{1/2} \ln |x-y| \, (\rho_n(x, y) - \rho_n(x) \, \rho_n(y)) \, dx \, dy$$

$$-\frac{n-1}{2n} \int_{-1/2}^{1/2} \ln |x-y| \, (\rho_n(x) - c(x))(\rho_n(y) - c(y)) \, dx \, dy$$

$$+\frac{1}{2n} \int_{-1/2}^{1/2} \ln |x-y| \, (\rho_n(x) - c(x)) \, c(y) \, dx \, dy$$

To obtain the expression for R_a, we have to replace $\rho_n(x)$ and $\rho_n(x, y)$ in this formula by $\rho_a(x)$ and $\rho_a(x, y)$, which are the correlation functions of the approximating Hamiltonian (2.8):

$$\rho_a(x) = \frac{\exp\{-nv(x) + n \int_{-1/2}^{1/2} \ln |x-y| \, c(y) \, dy\}}{\int_{-1/2}^{1/2} \exp\{-nv(x) + n \int_{-1/2}^{1/2} \ln |x-y| \, c(y) \, dy\} \, dx} \tag{2.11}$$

and $\rho_a(x, y) = \rho_a(x) \, \rho_a(y)$. Let us set $c(x) = c_n(x)$, where $c_n(x)$ is the unique solution of the "molecular field" equation:

$$c_n(x) = \rho_a(x) \tag{2.12}$$

[the existence and the uniqueness of solution (2.12) are proved in Lemma 3 of Section 3]. Then, by (2.11), $R_a(c_n) = 0$. Thus, we have proved the inequality $R \leqslant 0$. To estimate $R(c_n)$ from below, let us introduce the operators G, A, and A_n defined in $L_2(-1, 1)$ as follows:

$$(Gf)(x) = \int_{-1/2}^{1/2} (\rho_n(x, y) - \rho_n(x)\, \rho_n(y))\, f(y)\, dy$$

$$(Af)(x) = -\int_{-1/2}^{1/2} \ln |x - y|\, f(y)\, dy \tag{2.13}$$

$$(A_n f)(x) = -\int_{-1/2}^{1/2} a_n(x - y)\, f(y))\, dy$$

where the function $a_n(x)$, $x \in (-1, 1)$, is defined by the formula

$$a_n(x) = \begin{cases} \ln |x| & \text{if} \quad |x| \geqslant n^{-\alpha-3} \\ (\alpha+3) \ln n - n^{\alpha+3}(|x| - n^{-\alpha-3}) & |x| \leqslant n^{-\alpha-3} \end{cases} \tag{2.14}$$

with $\alpha = 2/\gamma$, γ being the Hölder constant of the function $V(\lambda)$. In terms of the operators (2.13), $R_n(c_n)$ can be expressed as

$$\begin{aligned} R_n(c_n) &= \frac{n-1}{2n} \operatorname{Tr} AG + \frac{n-1}{2n} (A(\rho_n - c_n), (\rho_n - c_n)) - \frac{1}{2n} (A(\rho_n - c_n), c_n) \\ &= \frac{n-1}{2n} \operatorname{Tr} A_n G + \frac{n-1}{2n} (A(\rho_n - c_n), (\rho_n - c_n)) \\ &\quad - \frac{1}{2n} (A(\rho_n - c_n), c_n) + \frac{n-1}{2n} \operatorname{Tr}(A - A_n) G \end{aligned} \tag{2.15}$$

On the basis of Lemma 4 of Section 3 one can conclude that

$$\left| \frac{1}{2} \operatorname{Tr}(A - A_n) G \right| \leqslant \text{const} \cdot \frac{\ln n}{n} \tag{2.16}$$

Besides, if we consider the operator G_n defined in $L_2(-1, 1)$ as

$$(G_n f)(x) = (Gf)(x) + \frac{1}{n-1} \rho_n(x)\, f(x) \tag{2.17}$$

then G_n is a positive operator. Indeed

$$\begin{aligned} (G_n f, f) = &\frac{1}{Z_n n(n-1)} \int_{-1/2}^{1/2} \left[\sum_{i=1}^{n} (f(x_i) - \langle f(x_i) \rangle) \right]^2 \\ &\times \exp\{ -nH(x_1, \dots, x_n) \}\, dx_1 \cdots dx_n \\ &+ \frac{1}{n(n-1)} \sum_{i=1}^{n} \langle f(x_i) \rangle^2 \geqslant 0 \end{aligned} \tag{2.18}$$

where $\langle f(x)\rangle = \int f(x)\, \rho_n(x)\, dx$. According to Lemma 2, the operator A_n is also positive and so we have

$$0 < \operatorname{Tr} A_n G_n = \operatorname{Tr} A_n G + \frac{1}{n-1}\, a_n(0) \tag{2.19}$$

Finally, on the basis of relations (2.10)–(2.19) we obtain

$$0 \leqslant \operatorname{Tr} A_n G_n + \frac{n-1}{2n}\,(A(\rho_n - c_n), (\rho_n - c_n)) \leqslant \text{const} \cdot \frac{\ln n}{n} \tag{2.20}$$

According to Lemma 2 of Section 3 and (2.18), both terms in the r.h.s. of (2.20) are nonnegative. Thus each of them admits the same bound. Now we are ready to estimate $d_n(z)$ of (2.2). Since it is easy to see that the second sum in the right-hand side of (2.3) is bounded from above by $n^{-1}(\Im z)^{-2}$, it suffices to estimate the expression

$$D(z) = \int_{-1/2}^{1/2} G_n(x, y)\, \frac{dx}{z-x} \cdot \frac{dy}{\bar{z}-y} \tag{2.21}$$

for any z, $\Im z \neq 0$. Introduce the projection operator P_z defined in $L_2(-1/2, 1/2)$:

$$(P_z f)(x) = \frac{1}{z-x} \int_{-1/2}^{1/2} f(y)\, \frac{dy}{\bar{z}-y} \tag{2.22}$$

Then $D(z) = \operatorname{Tr} P_z G_n$. Let us show that

$$P_z \leqslant \text{const} \cdot A_n \tag{2.23}$$

To this end, consider the function $q(x) \in L_2(-1, 1)$:

$$q(x) = \begin{cases} 2(x+1)(z+\frac{1}{2})^{-1} & \text{if } \; -1 < x \leqslant -\frac{1}{2} \\ (z-x)^{-1} & \text{if } \; -\frac{1}{2} < x \leqslant \frac{1}{2} \\ 2(1-x)(z-\frac{1}{2})^{-1} & \text{if } \; \frac{1}{2} < x < 1 \end{cases}$$

and $q(x+2k) = q(x)$, $k = 1, 2, \ldots$. This function is 2-periodic and continuous and its first derivative is bounded. Therefore its Fourier coefficients q_k satisfy the inequality

$$\sum_{k=0}^{\infty} k^2\, |q_k|^2 \leqslant \int_{-1}^{1} |q'(x)|^2\, dx \leqslant (\Im z)^{-3} + (\Im z)^{-2}$$

Consider the operator B_n defined in $L_2(-1, 1)$ by formula (3.11) below with $l(x) = -a_n(x)$, where $a_n(x)$ is defined in (2.14). Let $a_n^{(k)}$ be the corresponding Fourier coefficients. It follows from Lemma 2 and (2.14) that

$$\begin{aligned}(B_n^{-1}q, q)_2 &= 2\sum_{k=0}^{\infty} (a_n^{(k)})^{-1} |q_k|^2 \\ &\leqslant 2\sum_{k=0}^{\infty} (a_0^{(k)})^{-1} |q_k|^2 \\ &\leqslant \text{const}\cdot\left(|q_0|^2 + \sum_{k=1}^{\infty} k^2 |q_k|^2\right) \leqslant \text{const}\end{aligned} \tag{2.24}$$

where the symbol $(\cdot, \cdot)_2$ denotes the scalar product in $L_2(-1, 1)$. Let us take any function $f \in L_2(-1/2, 1/2)$ and consider its continuation $\hat{f}$ of the form (3.13). Then (2.24) implies

$$\begin{aligned}(P_z f, f) &= |(\hat{f}, q)|^2 = |(B_n^{1/2}\hat{f}, B_n^{-1/2}q)_2|^2 \leqslant (B_n\hat{f}, \hat{f})_2\,(B_n^{-1}q, q)_2 \\ &\leqslant \text{const}\cdot(A_n f, f)\end{aligned}$$

We have proved (2.23) and now on the basis of (2.20) we obtain

$$D(z) = \operatorname{Tr} G_n P_z \leqslant \text{const}\cdot\operatorname{Tr} G_n A_n \leqslant \text{const}\cdot\frac{\ln n}{n} \tag{2.25}$$

As mentioned above, this estimate proves (2.2), i.e., the self-averaging property of $g_n(z)$.

Let us consider now $\rho(x) = \lim_{n\to\infty} c_n(x)$. Lemma 3 guarantees that this limit exists and is uniquely determined by the relations (1.8)–(1.12). To prove Theorem 1, it is enough now to prove that

$$(P_z(\rho_n - \rho), \rho_n - \rho) \to 0 \qquad \text{as} \quad n \to \infty \tag{2.26}$$

Since from Lemma 2 it follows that $A_n \leqslant A$, we get from (2.23) the inequality $P_z \leqslant \text{const}\cdot A$. Thus (2.26) follows from (2.20) and Lemma 2. Theorem 1 is proved.

Let us reformulate the result of Theorem 1 in terms of statistical mechanics.

Proposition. Let us consider an n-particle one-dimensional statistical mechanical system specified by the Gibbs distribution

$$p_n(\lambda_1,\dots,\lambda_n) = Q_n^{-1}\exp\{-nH_n(\lambda_1,\dots,\lambda_n)\} \tag{2.27}$$

where the Hamiltonian H_n is given by (1.6). Then, in the thermodynamic limit $n=\infty$, the ground-state energy per site is (1.15), the first correlation function satisfies the self-consistent equation (1.12), and all the higher correlation functions are products of the first one.

Proof. Inequality (2.10) together with (2.11), (2.15), (2.16), and (2.20) implies that there exists

$$E_1 = -\lim_{n\to\infty} \frac{1}{n^2} \ln Z_n = -\lim_{n\to\infty} \Psi_n(c_n)$$

Therefore

$$E_1 = -\tfrac{1}{2} \int_{\operatorname{supp}\rho} \ln|x-y|\, \rho(x)\, \rho(y)\, dx\, dy + \int_{\operatorname{supp}\rho} v(x)\, \rho(x)\, dx$$

The other assertions of the Proposition follows from inequality (2.25) supplemented by the fact that, according to Lemma 2 of Section 3, the operators A, A_n are positive.

Remark. For the unitary invariant analog of (1.1), i.e., for $\beta=2$ in (1.5), the self-averaging property (2.2) and the resulting weak factorization of $\rho_n(\lambda_1, \lambda_2)$ into the product $\rho_n(\lambda_1)\, \rho_n(\lambda_2)$ can be easily proven by using the orthogonal polynomial technique,[2] according to which

$$\begin{aligned} \rho_n(\lambda_1,..., \lambda_k) &= \int p_n(\lambda_1,..., \lambda_n)\, d\lambda_{k+1} \cdots d\lambda_n \\ &= \frac{n^k}{n(n-1)\cdots(n-k+1)} \det \|K_n(\lambda_i, \lambda_j)\|_{i,j=1}^k \end{aligned} \tag{2.28}$$

where

$$\begin{aligned} K_n(\lambda, \mu) &= \frac{1}{n} \sum_{l=1}^{n} \psi_l^{(n)}(\lambda)\, \psi_l^{(n)}(\mu) \\ \psi_l^{(n)}(\lambda) &= \exp\left\{-\frac{n}{2} V(\lambda)\right\} P_l^{(n)}(\lambda) \end{aligned} \tag{2.29}$$

and $P_l^{(n)}$, $l=0, 1,...$, are the orthonormal polynomials with respect to the weight $\exp\{-nV(\lambda)\}$. In particular,

$$\begin{aligned} \rho(\lambda) &= \frac{1}{n} K_n(\lambda, \lambda) \\ \rho_n(\lambda_1, \lambda_2) &= \frac{n}{n-1} (\rho_n(\lambda_1)\, \rho_n(\lambda_2) - K_n^2(\lambda_1, \lambda_2)) \end{aligned} \tag{2.30}$$

These relations yield

$$d_n(z)=\int\frac{K_n(\lambda,\lambda)\,d\lambda}{|z-\lambda|^2}-\int\frac{K_n^2(\lambda_1,\lambda_2)\,d\lambda_1\,d\lambda_2}{(z-\lambda_1)(\bar{z}-\lambda_2)}$$

According to the orthonormality of the system $\{\psi_l^{(n)}\}_{l=1}^{\infty}$,

$$\int K_n(\lambda,\lambda)\,d\lambda=1,\qquad \int K_n^2(\lambda_1,\lambda_2)\,d\lambda_1\,d\lambda_2=\frac{1}{n}$$

Using these relations and the inequality $|z-\lambda|^{-1}\leqslant|\Im z|^{-1}$, we obtain that

$$d_n(z)\leqslant\frac{2}{n\,|\Im z|^2}\tag{2.31}$$

and somewhat more detailed arguments based on the same technique yield that $d_n(z)=O(n^{-2})$. These relations [cf. (2.25)] imply the weak self-averaging property of the IDS, i.e., the factorization property of the respective measures. The pointwise factorization properties can be proven based directly on (2.30) and on the Christoffel–Darboux formula from the theory of orthogonal polynomials,

$$K_n(\lambda,\mu)=\frac{1}{n}\cdot\frac{\psi_n^{(n)}(\lambda)\,\psi_{n-1}^{(n)}(\mu)-\psi_n^{(n)}(\mu)\,\psi_{n-1}^{(n)}(\lambda)}{\lambda-\mu}\tag{2.32}$$

Thus, if $\psi_l^{(n)}(\lambda)$ is bounded for any fixed λ, all n, and $l=n$, $n-1$, then (2.30) and (2.32) imply the factorization of $\rho_n(\lambda_1,\lambda_2)$ into the product $\rho_n(\lambda_1)\,\rho_n(\lambda_2)$ for $\lambda_1=\lambda_2$ and $n\to\infty$ (and in fact all the higher correlation functions for distinct arguments). The proof of the same property for coinciding arguments is based on the analog of (2.32) for $\lambda=\mu$, which contains the derivatives of $\psi_l^{(n)}(\lambda)$ for $l=n$, $n-1$ in its r.h.s. Thus, in this case we need boundedness of the respective derivatives uniform in n.

3. AUXILIARY RESULTS

Lemma 1. If a function $V(\lambda)$ satisfies condition (1.2), then there exists a positive number L such that if we define $\rho_{nL}(\lambda_1,\ldots,\lambda_m)$ as

$$\begin{aligned}\rho_{nL}(\lambda_1,\ldots,\lambda_m)&=(Z_{nL})^{-1}\int\prod_{i=m+1}^{n}\chi_L(\lambda_i)\,d\lambda_i\exp\{-nH_n(\lambda_1,\ldots,\lambda_n)\}\\ Z_{nL}&=\int\prod_{i=1}^{n}\chi_L(\lambda_i)\,d\lambda_i\exp\{-nH_n(\lambda_1,\ldots,\lambda_n)\}\end{aligned}\tag{3.1}$$

where $\chi_L(\lambda)$ is the characteristic function of the interval $(-L, L)$, then for $|\lambda_1|,..., |\lambda_m| \leqslant L$,

$$|\rho_n(\lambda_1,..., \lambda_m) - \rho_{nL}(\lambda_1,..., \lambda_m)| < \rho_{nL}(\lambda_1,..., \lambda_m)\, e^{-Cn} \tag{3.2}$$

and for $|\lambda_1|,..., |\lambda_j| \geqslant L$, $|\lambda_{j+1}|,..., |\lambda_m| \leqslant L$,

$$|\rho_n(\lambda_1,..., \lambda_m)| \leqslant \exp\left\{-nC' \sum_{k=1}^{j} \ln |\lambda_k|\right\} \tag{3.3}$$

where C and C' do not depend on n and m.

Proof. Let us choose $L > 2e$ to satisfy the inequality

$$V(\lambda) - \max_{|\lambda'| < e} V(\lambda') > \left(2 + \frac{\varepsilon}{2}\right) \ln |\lambda|, \qquad |\lambda| > L \tag{3.4}$$

and represent $\rho_n(\lambda_1,..., \lambda_m)$ as follows:

$$\begin{aligned} \rho_n(\lambda_1,..., \lambda_m) &= Z_n^{-1} \sum_{k=0}^{n-m} \binom{n-m}{k} \int I_k(\lambda) \prod_{i=m+1}^{n} d\lambda_i \\ Z_n &= \sum_{k=0}^{n} \binom{n}{k} \int I_k(\lambda) \prod_{i=m+1}^{n} d\lambda_i \end{aligned} \tag{3.5}$$

where

$$I_k(\lambda) = \prod_{i=1}^{n-k} \chi_L(\lambda_i) \prod_{i=n-k+1}^{n} [1 - \chi_L(\lambda_i)] \exp\{-nH_n(\lambda_1,..., \lambda_n)\}$$

Then we can write

$$\begin{aligned} &\int I_k(\lambda) \prod_{i=n-k+1}^{n} d\lambda_i \\ &= \sum_{j_1,...,j_k = \pm 1, \pm 2,...} \int I_0(\lambda_1,..., \lambda_{n-k}, \lambda_{n-k+1} + 2j_1 L,..., \lambda_n + 2j_k L) \prod_{i=n-k+1}^{n} d\lambda_i \end{aligned} \tag{3.6}$$

Now we will estimate

$$\begin{aligned} \Delta = &\ln \int I_0(\lambda_1,..., \lambda_{n-k}, \lambda'_{n-k+1} + 2j_1 L,..., \lambda'_n + 2j_k L) \prod_{i=n-k+1}^{n} d\lambda'_i \\ &- \ln \int I_0(\lambda_1,..., \lambda_{n-k}, \lambda'_{n-k+1},..., \lambda'_n) \prod_{i=n-k+1}^{n} d\lambda'_i \end{aligned} \tag{3.7}$$

To this end, we divide the interval $(-e, e)$ into n^2 equal intervals by the points $y_1,..., y_{n^2-1}$. Now for any fixed configuration $\{\lambda_1,..., \lambda_{n-k}\}$ we exclude y's which are the nearest neighbors of $\{\lambda_i\}$ and consider $2en^{-2} \sum_{y_i} \ln |y_i - \lambda_r|$. Since for all r,

$$2en^{-2} \sum_{y_i} \ln |y_i - \lambda_r| > \int_{-e}^{e} \ln |y - \lambda_r| \, dy + O(n^{-2}) \geqslant O(n^{-2})$$

we have

$$2en^{-2} \sum_{y_i} \sum_{r=1}^{n-k} \ln |y_i - \lambda_r| > (n-k) \, O(n^{-2})$$

Thus, there exists y_i such that

$$\sum_{r=1}^{n-k} \ln |y_i - \lambda_r| > (n-k) \, O(n^{-2})$$

Let us set $\lambda_{n-k+1} = y_i$ and repeat the above procedure for $\{\lambda_1,..., \lambda_{n-k+1}\}$. Then, we find λ_{n-k+2} such that

$$\sum_{r=1}^{n-k+1} \ln |y_i - \lambda_r| > (n-k+1) \, O(n^{-2})$$

After k steps, we will have the configuration $\lambda_1,..., \lambda_n$ such that

$$|\lambda_{n-k+p} - \lambda_r| \geqslant 2en^{-2}, \qquad r, p > 0 \tag{3.8}$$

and

$$\sum_{r=1}^{n-k+p} \ln |y_i - \lambda_r| > O(n^{-1}) \tag{3.9}$$

Thus,

$$\sum_{p=1}^{k} \sum_{r=1}^{n-k+p} \ln |y_i - \lambda_r| > knO(n^{-2}) \tag{3.10}$$

Let us change the variables in the first term of the r.h.s. of (3.7):

$$\int_{-L}^{L} I_0(\lambda_1,..., \lambda_{n-k}, \lambda'_{n-k+1} + 2j_1 L,..., \lambda'_n + 2j_k L) \prod_{i=n-k+1}^{n} d\lambda'_i$$

$$= (Ln^3)^k \int_{-n^{-3}}^{n^{-3}} I_0(\lambda_1,..., \lambda_{n-k}, Ln^3 t_1 + 2j_1 L,..., Ln^3 t_k + 2j_k L) \prod_{i=1}^{k} dt_i$$

We obtain that $\Delta \leqslant k \ln Ln^3 + \Delta_1$, where

$$\Delta_1 = n \max_t \{H_n(\lambda_1,..., \lambda_{n-k}, Ln^3 t_1 + 2j_1 L,..., Ln^3 t_k + 2j_k L) - H_n(\lambda_1,..., \lambda_{n-k}, \lambda_{n-k+1} + t_1,..., \lambda_n + t_k)\}$$

and $\lambda_{n-k+1},..., \lambda_n$ are chosen by the above procedure and satisfy (3.8) and (3.9). Thus,

$$\begin{aligned}\Delta_1 &\leqslant -n \sum_{p=1}^{k} [V((2|j_p|-1)L) - \max_{[-e,e]} V(\lambda')] \\ &\quad + \sum_{p=1}^{k} \sum_{r=1}^{n-k+p} \ln(2|j_p|+2)L - \sum_{p=1}^{k} \sum_{r=1}^{n-k+p} \ln|y_i - \lambda_r| \\ &\leqslant -\varepsilon n/2 \sum_{p=1}^{k} \sum_{r=1}^{n-k+p} \ln(2|j_p|-1)L + knO(n^{-2})\end{aligned}$$

and, as a result,

$$\begin{aligned}&\int I_k(\lambda) \prod_{i=n-k+1}^{n} d\lambda_i \\ &\quad \leqslant (Ln^3)^k \left[\sum_{j=1}^{\infty} 2((2j-1)L)^{-n\varepsilon/2)}\right]^k \int I_0(\lambda) \prod_{i=n-k+1}^{n} d\lambda_i \\ &\quad \leqslant \text{const} \cdot \int I_0(\lambda) \prod_{i=n-k+1}^{n} d\lambda_i \exp\left(\frac{-kn\varepsilon \ln L}{3}\right)\end{aligned}$$

Therefore, by using the representation (3.5) we obtain

$$\rho_n(\lambda_1,..., \lambda_m) = \rho_{nL}(\lambda_1,..., \lambda_m)(1 + O(e^{-nM}))$$

Inequality (3.3) can be proved similarly.

Lemma 2. If the operators A, A_n are defined in $L_2(-1/2, 1/2)$ by relations (2.13) and (2.14), then

$$0 \leqslant A_0 \leqslant A_n \leqslant A$$

where the operator A_0 is defined by relation (2.13) with $a_0(x) = 1 - |x|$.

Proof. It is enough to prove that for any convex positive function $l(x)$ defined on the interval $(0, 1/2)$ the operator $A^{(l)}$ defined in $L_2(-1/2, 1/2)$ by the formula

$$(A^{(l)}f)(x) = \int_{-1/2}^{1/2} l(|x-y|)\, dy$$

is positive. Consider the operator B defined in $L_2(-1, 1)$ as follows:

$$(Bf)(x) = \int_{-1}^{1} \hat{l}(x-y)\, f(y)\, dy \tag{3.11}$$

where $\hat{l}(x)$ is a 2-periodic continuation of $l(|x|)$ defined in the interval $(-1, 1)$. As is well known, the operators of this form have a spectrum which consists of the Fourier coefficients of the periodic function $\hat{l}(x)$. Therefore the positivity of B is a consequence of the inequalities

$$l^{(k)} = \int_{-1}^{1} l(|x|) \cos \pi k x\, dx > 0, \qquad k = 0, 1,... \tag{3.12}$$

For $k = 0$,

$$l^{(0)} = -\int_{-1}^{1} l(|x|)\, dx > 0$$

For $k > 0$,

$$l^{(k)} = 2\int_0^1 l(x) \cos(\pi k x)\, dx = -\frac{2}{\pi k}\int_0^1 \sin(\pi k x)\, l'(x)\, dx$$

$$= -\int_0^{1/k} \sin(\pi k x)\, [l'(x) - l'(x + 1/k) + l'(x + 2/k) + \cdots]\, dx$$

But since $l'(x)$ is an increasing function for $x > 0$, we conclude that $l^{(k)} > 0$ and therefore B is positive. Now, for any function $f \in L_2(-1/2, 1/2)$, let us consider the function $\hat{f} \in L_2(-1, 1)$ such that

$$\hat{f} = \begin{cases} f(x) & \text{if } |x| \leqslant 1/2 \\ 0 & \text{otherwise} \end{cases} \tag{3.13}$$

Then it is evident that $(A^{(l)}f, f) = (B\hat{f}, \hat{f})_1 > 0$, where we denoted by $(\cdot, \cdot)_1$ the scalar product in $L_2(-1, 1)$. Lemma 2 is proved.

Lemma 3. Let the functional $\Phi_n(c)$ be defined on the space L^* of the functions satisfying inequality (2.7) by the formulas [cf. (2.9)]

$$\Phi_n(c) = -\frac{1}{2}\int_{-1/2}^{1/2} \ln |x-y|\, c(x)\, c(y)\, dx\, dy$$

$$+ \frac{1}{n} \ln \int_{-1/2}^{1/2} dx \exp\left\{-nv(x) + n\int_{-1/2}^{1/2} \ln |x-y|\, c(y)\, dy\right\}$$

Then:

(i) $\Phi_n(c)$ has the unique point of the extremum, c_n, which is the solution of equation (2.12).

(ii) There exists $\rho = L^* - \lim_{n \to \infty} c_n$.

(iii) $\rho(x)$ satisfies relations (1.7)–(1.11) and these relations determine ρ uniquely.

Proof. (i) Since the operator A specified by (2.13) is positive, it is easy to see that $\Phi_n(c)$ is convex, i.e.,

$$\Phi_n\left(\frac{c_1+c_2}{2}\right) \leqslant \frac{\Phi_n(c_1)+\Phi_n(c_2)}{2}$$

Besides, on the basis of the Jensen inequality we have

$$\frac{1}{n}\ln\int_{-1/2}^{1/2} dx \exp\left\{-nv(x)+2n\int_{-1/2}^{1/2} dy \ln|x-y|\, c(y)\right\}$$
$$\leqslant -\int_{-1/2}^{1/2} v(x)\, dx + \int_{-1/2}^{1/2} \ln|x-y|\, c(y)\, dx\, dy$$

and

$$\Phi_n(c) \geqslant \tfrac{1}{2}(Ac, c) - \int_{-1/2}^{1/2} (Ac)(x)\, dx - \int_{-1/2}^{1/2} v(x)\, dx$$
$$\geqslant \tfrac{1}{2}\int_{-1/2}^{1/2} \ln|x-y|\, dx\, dy - \int_{-1/2}^{1/2} v(x)\, dx$$

Therefore $\Phi_n(c)$ is bounded below. Consider a minimizing sequence $\{c^{(k)}(x)\}$ such that

$$\lim_{k\to\infty} \Phi_n(c^{(k)}) = \inf_{c \in L^*} \Phi_n(c) \equiv \Phi_n^*$$

Then, for any $\varepsilon > 0$,

$$\Phi_n^* + \varepsilon > \Phi_n(c^{(k)}) > \Phi_n^*$$

if k is large enough. For these k and m,

$$\Phi_n^* + \varepsilon > \frac{\Phi_n(c^{(k)})+\Phi_n(c^{(m)})}{2} > \Phi_n\left(\frac{c^{(k)}+c^{(m)}}{2}\right) > \Phi_n^*$$

Thus,

$$(A(c^{(k)}-c^{(m)}), (c^{(k)}-c^{(m)})) \leqslant 8\left(\frac{\Phi_n(c^{(k)})+\Phi_n(c^{(m)})}{2}-\Phi_n\left(\frac{c^{(k)}+c^{(m)}}{2}\right)\right)\leqslant 4\varepsilon \tag{3.14}$$

In other words, the sequence $\{c^{(k)}\}$ satisfies the Cauchy conditions in the Hilbert space L^* with a scalar product $(\cdot,\cdot)_* = (A\cdot,\cdot)$ and, as a result, has the limit c_n in this space. This point c_n is a point of the extremum for Φ_n. Besides, since the second derivative of Φ_n in any direction is bounded below by 1, then c_n is a unique extremum point. Now, taking the first derivative of Φ_n at the point c_n, we obtain that c_n satisfies Eq. (2.12).

(ii) According to the Hölder inequality,

$$\left(\int_{-1/2}^{1/2} \exp\{nu(x)\}\, dx\right)^{1/n} \leqslant \left(\int_{-1/2}^{1/2} \exp\{(n+1)\, u(x)\}\, dx\right)^{1/(n+1)}$$

Therefore for any c, $\Phi_n(c) \leqslant \Phi_{n+1}(c)$, and thus

$$\Phi_n(c_n) \leqslant \Phi_{n+1}(c_{n+1})$$

Besides, it is easy to see that the sequence of numbers $\Phi_n(c_n)$ is bounded from above and therefore from the latter inequality it follows that for any $\varepsilon \geqslant 0$ there exists a number n such that for any $m \geqslant n$,

$$0 \leqslant \Phi_m(c_m) - \Phi_n(c_n) \leqslant \varepsilon$$

On the other hand, since c_n is the minimum point of Φ_n,

$$0 \leqslant \Phi_n(c_m) - \Phi_n^* \leqslant \Phi_m(c_m) - \Phi_n(c_n) \leqslant \varepsilon$$

Thus, repeating the arguments (3.14), we get

$$(A(c_n - c_m), (c_n - c_m)) \leqslant 4\varepsilon$$

This inequality implies that the sequence c_n of the minimum points is fundamental in L^* and therefore has the limiting point ρ in this space.

(iii) Consider

$$u_n(x) = \int_{-1/2}^{1/2} \ln|x-y|\, c_n(y)\, dy - v(x)$$

Since

$$\lim_{n\to\infty} (n)^{-1} \ln \int_{-1/2}^{1/2} \exp\{nu_n\}\, dx < \infty$$

we conclude that the function

$$u(x) = \lim_{n \to \infty} u_n(x) = \int_{-1/2}^{1/2} \ln |x-y| \, \rho(y) \, dy - v(x)$$

is bounded above. Consider $u^* = \max u(x)$. It is easy to see that if for some x, $u(x) < u^*$, then

$$\rho(x) = \lim_{n \to \infty} c_n(x) = \lim_{n \to \infty} \frac{\exp\{nu_n\}}{\int_{-1/2}^{1/2} \exp\{nu_n\} \, dx} = 0$$

Thus, $\operatorname{supp} \rho(x) \subset \{x: u(x) = u^*\}$. Therefore $\rho(x)$ satisfies conditions (1.8)–(1.12). Suppose that there exists another function $\rho_1(x)$ satisfying these conditions. Let

$$u^1(x) = \int_{-1/2}^{1/2} \ln |x-y| \, \rho_1(y) \, dy - v(x)$$

Consider

$$r_n = \rho_1 + n^{-1/2}(c_n - \rho_1)$$

$$u_n^1(x) = \int_{-1/2}^{1/2} \ln |x-y| \, r_n(y) \, dy - v(x) = u^1 + n^{-1/2}(u_n - u^1)$$

$$= u^1 + n^{-1/2}(u - u^1) + n^{-1/2}(u_n - u)$$

Since Φ_n is a convex functional, then

$$0 \leqslant \Phi_n(r_n) - \Phi_n(c_n) \leqslant -(r_n, u_n^1 - u_n) + \frac{\int_{-1/2}^{1/2} (u_n^1 - u_n) \exp\{nu^1\} \, dx}{\int_{-1/2}^{1/2} \exp\{nu_n^1\} \, dx} \tag{3.15}$$

Besides, since $c_n \to \rho$, the r.h.s of (3.15), S_n, can be rewritten as

$$S_n = -(\rho_1, u^1 - u)$$

$$+ \frac{\int_{-1/2}^{1/2} (u^1 - u) \exp\{nu^1 + \sqrt{n}\,(u - u^1) + o(\sqrt{n})\} \, dx}{\int_{-1/2}^{1/2} \exp\{nu^1 + \sqrt{n}\,(u - u^1) + o(\sqrt{n})\} \, dx} + o(1)$$

$$\to -(\rho_1, u^1 - u) + \min_{u^1(x) = \max u^1} (u^1 - u) + o(1)$$

Since $\operatorname{supp} \rho_1 \subset \{u^1(x) = \max u^1\}$ and ρ_1 satisfies conditions (1.8), (1.9), then

$$(\rho_1, u^1 - u) > \min(u^1 - u)$$

and we have $S_n = o(1)$, $n \to \infty$, and as in (3.14) we get

$$(\rho_1 - \rho, \rho_1 - \rho)_* = (r_n - c_n, r_n - c_n) + o(1) \to 0, \qquad n \to \infty$$

Therefore $\rho_1 = \rho$.

Lemma 3 is proved.

Lemma 4. If the operators A, A_n, and G defined in $L_2(-1, 1)$ are specified by formulas (2.13) and (2.14), then

$$|\operatorname{Tr}(A - A_n) G| \leqslant \text{const} \cdot \frac{\ln n}{n} \tag{3.16}$$

Proof. By the definition (2.13),

$$\begin{aligned}
&|\operatorname{Tr}(A - A_n) G| \\
&\quad = \left| \int_{-1/2}^{1/2} dx\, dy\, [\ln |x - y| - a_n(x - y)][\rho_n(x, y) - \rho_n(x)\, \rho_n(y)] \right| \\
&\quad = \left| \int_{-1/2}^{1/2} dx\, dy\, [\ln |x - y| - a_n(x - y)] \right. \\
&\qquad \left. \times [\rho_n(x, y) - \rho_n(x)\, \rho_n(y)]\, \psi_n(x - y) \right| \\
&\quad \leqslant \int_{-1/2}^{1/2} dx\, dy\, (|\ln |x - y|| + \text{const} \cdot \ln n) \\
&\qquad \times [\rho_n(x, y) + \rho_n(x)\, \rho_n(y)]\, \psi_n(x - y)
\end{aligned} \tag{3.17}$$

where $\psi_n(x)$ is the characteristic function of the interval $(-n^{-3-\alpha}, n^{-3-\alpha})$. Let us now estimate

$$r_n = \int \psi_n(x - y)\, \rho_n(x, y)\, |\ln |x - y||\, dx\, dy$$

To this end, we rewrite r_n as follows:

$$r_n = \frac{\int |\ln |x - y|| \cdot |x - y|\, \phi(x, y)\, \psi_n(x, y)\, dx\, dy}{\int |x - y|\, \phi(x, y)\, dx\, dy} \tag{3.18}$$

where

$$\phi(x, y) = \int \exp\left\{-nv(x) - nv(y) - n \sum_{i=3}^{n} v(x_i) + \sum_{i=3}^{n} \ln|x - x_i| + \sum_{i=3}^{n} \ln|y - x_i| + \sum_{3 \leqslant i < j}^{n} \ln|x_i - x_j|\right\} dx_3 \cdots dx_n$$
$$= \int_{-1/2}^{1/2} \exp\{-nH^1(x, y, x_3,..., x_n)\}\, dx_3 \cdots dx_n \qquad (3.19)$$

Let us change the variables in the integral (3.18):

$$x' = x - n^{-\alpha}, \qquad y' = y + n^{-\alpha}$$
$$x_i' = \begin{cases} x_i - n^{-\alpha} & \text{if } x_i \leqslant \frac{1}{2}(x+y) \\ x_i + n^{-\alpha} & \text{if } x_i > \frac{1}{2}(x+y) \end{cases}$$

Since $V(x)$ satisfies the Hölder condition (1.2c) and $\alpha = 2/\gamma$, then

$$|v(x_i) - v(x_i')| = |V(Lx_i) - V(Lx_i')| \leqslant \text{const} \cdot \frac{1}{n^{\alpha\gamma}} = \text{const} \cdot \frac{1}{n^2}$$

Besides, under this change of variables all the differences will either be the same or increase. Therefore,

$$H^1(x, y,..., x_n) \leqslant H^1(x', y',..., x_n') + \text{const} \cdot \frac{1}{n}$$

and we obtain the inequality

$$\phi(x, y) \leqslant \text{const} \cdot \phi(x', y')$$

Inserting this esitmate into (3.19), we get

$$r_n \leqslant \text{const} \cdot \frac{n^{-1-\alpha} \ln n \int_{|x'-y'| \geqslant n^{-\alpha}} \phi(x', y')\, dx'\, dy'}{\int_{|x'-y'| \geqslant n^{-\alpha}} \phi(x', y')\, |x' - y'|\, dx'\, dy'}$$
$$\leqslant \text{const} \cdot \frac{n^{-1-\alpha} \ln n \int_{|x'-y'| \geqslant n^{-\alpha}} \phi(x', y')\, dx'\, dy'}{\int_{|x'-y'| \geqslant n^{-\alpha}} n^{-\alpha} \phi(x', y')\, dx'\, dy'} = \text{const} \cdot \frac{\ln n}{n}$$

By the same arguments, we can prove also that

$$r_n' = \int \psi_n(x-y)\, \rho_n(x)\, \rho_n(y)\, |\ln|x-y||\, dx\, dy \leqslant \text{const} \cdot \frac{\ln n}{n}$$

$$r_n'' = \int \psi_n(x-y)\, \rho_n(x)\, \rho_n(y)\, dx\, dy \leqslant \text{const} \cdot \frac{1}{n}$$

Finally, from (3.18), we obtain (3.16).

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REFERENCES

1. E. Wigner, Statistical properties of real symmetric matrices with many dimensions, *Can. Math. Proc.* **1957**:174–198.
2. M. L. Mehta, *Random Matrices* (Academic Press, New York, 1967).
3. V. L. Girko, *Spectral Theory of Random Matrices* (Kluwer, Dordrecht, 1990).
4. F. Haake, *Quantum Signatures of Chaos* (Springer-Verlag, Heidelberg, 1991).
5. R. Fernandez, J. Frohlich, and A. Sokal, *Random Walks, Critical Phenomena and Triviality in the Quantum Field Theory* (Springer-Verlag, Heidelberg, 1992).
6. E. Brezin, C. Itzykson, G. Parisi, and J. Zuber, Planar diagrams, *Commun. Math. Phys.* **59**:35–51 (1978).
7. D. Bessis, C. Itzykson, and J. Zuber, Quantum field theory techniques in graphical enumeration, *Adv. Appl. Math.* **1**:109–157 (1980).
8. D. Lechtenfeld, A. Ray, and R. Ray, Phase diagram and orthogonal polynomials in multiple-well matrix models, *Int. J. Mod. Phys.* **6**:4491–4515 (1991).
9. L. Pastur, On the universality of the level spacing distribution for some ensemble of random matrices, *Lett. Math. Phys.* **25**:259–265 (1992).
10. F. J. Dyson, Statistical theory of the energy levels of the complex system. I–III, *J. Math. Phys.* **3**:379–414 (1962).
11. M. Kac, G. Uhlenbeck, and P. Hemmer, On the van-der-Waals theory of vapor liquid equilibrium, *J. Math. Phys.* **4**:216–247 (1963).
12. P. Hemmer and J. Lebowitz, System with weak long-range potentials, in *Phase Transitions and Critical Phenomena* (Academic Press, New York, 1973).
13. L. Pastur and M. Shcherbina, Long-range limit of correlation functions of lattice systems, *Teor. Mat. Fiz.* **61**:3–16 (1984) [in Russian].
14. M. Shcherbina, Classical Heisenberg model at zero temperature, *Teor. Mat. Fiz.* **81**:134–144 (1989) [in Russian].
15. K. Demetrefi, Two-dimensional quantum gravity, matrix models and string theory, *Int. J. Mod. Phys. A* **8**:1185–1244 (1993).
16. E. Brezin and A. Zee, Universality of the correlations between eigenvalues of large random matrices, *Nucl. Phys. B* **402**:613–627 (1993).
17. G. M. Cicuta, L. Molinari, and E. Montaldi, Large *n* phase transition in low dimensions, *Mod. Phys. Lett.*.*A* **1**:125–129 (1986).
18. N. I. Muskhelishvili, *Singular Integral Equations* (Noordhoff, Groningen, 1953).